countable uncountable
digital analog
discrete continuous

4.3 Fourier Series

Definition 4.41. Exponential Fourier series: Let the (real or complex) signal $r(t)$ be a *periodic* signal with period T_0. $r(t+T_0) = r(t) \ \forall t$

Suppose the following *Dirichlet* conditions are satisfied:

(a) $r(t)$ is absolutely integrable over its period; i.e., $\int_0^{T_0} |r(t)|dt < \infty$.

(b) The number of maxima and minima of $r(t)$ in each period is finite.

(c) The number of discontinuities of $r(t)$ in each period is finite.

Then $r(t)$ can be "expanded" into a linear combination of the complex exponential signals $\left(e^{j2\pi(kf_0)t}\right)_{k=-\infty}^{\infty}$ as

f is discrete

$$\tilde{r}(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(kf_0)t} = c_0 + \sum_{k=1}^{\infty}\left(c_k e^{j2\pi(kf_0)t} + c_{-k}e^{-j2\pi(kf_0)t}\right) \tag{37}$$

where

$$f_0 = \frac{1}{T_0} \quad \text{and}$$

$$c_k = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} r(t)\, e^{-j2\pi(kf_0)t}dt, = \left\langle r(t) e^{-j2\pi k f_0 t}\right\rangle \tag{38}$$

for some *arbitrary* α. $c_0 = \langle r(t) \rangle$

We give some remarks here.

- $\tilde{r}(t) = \begin{cases} r(t), & \text{if } r(t) \text{ is continuous at } t \\ \frac{r(t^+)+r(t^-)}{2}, & \text{if } r(t) \text{ is not continuous at } t \end{cases}$

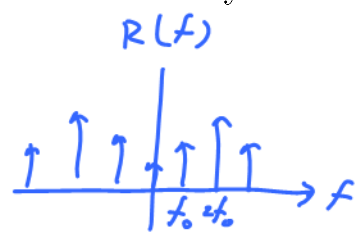
  Although $\tilde{r}(t)$ may not be exactly the same as $r(t)$, for the purpose of our class, it is sufficient to simply treat them as being the same (to avoid having two different notations). Of course, we need to keep in mind that unexpected results may arise at the discontinuity points.

- The parameter α in the limits of the integration (38) is arbitrary. It can be chosen to simplify computation of the integral. Some references simply write $c_k = \frac{1}{T_0}\int_{T_0} r(t)\, e^{-jk\omega_0 t}dt$ to emphasize that we only need to integrate over one period of the signal; the starting point is not important.

- The coefficients c_k are called the (k^{th}) ***Fourier (series) coefficients*** of (the signal) $r(t)$. These are, in general, complex numbers.
- $c_0 = \frac{1}{T_0}\int_{T_0} r(t)\,dt$ = average or DC value of $r(t)$
- The quantity $f_0 = \frac{1}{T_0}$ is called the ***fundamental frequency*** of the signal $r(t)$. The kth multiple of the fundamental frequency (for positive k's) is called the kth ***harmonic***.
- $c_k e^{j2\pi(kf_0)t} + c_{-k} e^{-j2\pi(kf_0)t}$ = the k^{th} ***harmonic component*** of $r(t)$. $k = 1 \Rightarrow$ ***fundamental component*** of $r(t)$.

4.42. Being able to write $r(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(kf_0)t}$ means we can easily find the Fourier transform of any periodic function:

$$r(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(kf_0)t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} R(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_0)$$



The Fourier transform of any periodic function is simply a bunch of weighted delta functions occuring at multiples of the fundamental frequency f_0.

4.43. Formula (38) for finding the Fourier (series) coefficients

③ Extra scaling by $\frac{1}{T_0}$

① The integration is over only one period of $r(t)$

$$c_k = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} r(t)\, e^{-j2\pi(kf_0)t} dt \tag{39}$$

② $f = kf_0$

is strikingly similar to formula (5) for finding the Fourier transform:

$$R(f) = \int_{-\infty}^{\infty} r(t) e^{-j2\pi f t} dt. \tag{40}$$

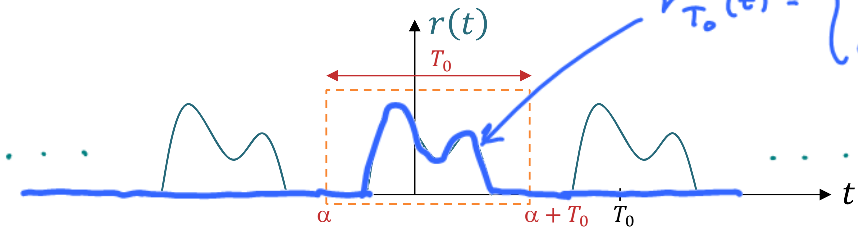
There are three main differences.

We have spent quite some effort learning about the Fourier transform of a signal and its properties. It would be nice to have a way to reuse those concepts with Fourier series. Identifying the three differences above lets us see their connection.

4.44. Getting the Fourier coefficients from the Fourier transform:

Step 1 Consider a restricted version $r_{T_0}(t)$ of $r(t)$ where we only consider $r(t)$ for one period.

$$r_{T_0}(t) = \begin{cases} r(t), & \alpha \le t < \alpha + T_0, \\ 0, & \text{otherwise.} \end{cases}$$



Step 2 Find the Fourier transform $R_{T_0}(f)$ of $r_{T_0}(t)$

Step 3 ***The Fourier coefficients are simply scaled samples of the Fourier transform***:

$$c_k = \frac{1}{T_0} R_{T_0}(kf_0).$$

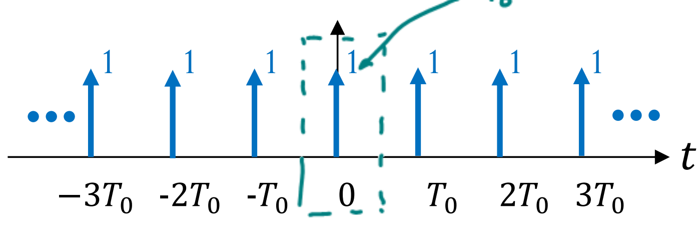
uniform pulse train

Example 4.45. Train of Impulses: Find the Fourier series expansion for the train of impulses Dirac comb

$$Ш^{(T_0)}(t) = \delta^{(T_0)}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(kf_0)t} = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{j2\pi(kf_0)t}$$

drawn in Figure 21. This infinite train of equally-spaced -functions is usually denoted by the Cyrillic letter (shah).

$$r_{T_0}(t) = \delta(t) \xrightarrow{\mathcal{F}} R_{T_0}(f) \equiv 1$$

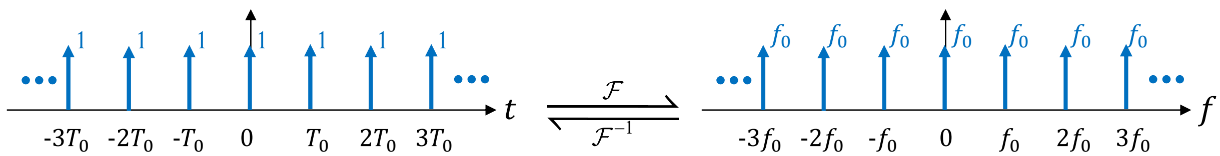


$$c_k = \frac{1}{T_0} R_{T_0}(kf_0) = \frac{1}{T_0}$$

Figure 21: Train of impulses

4.46. The Fourier series derived in Example 4.44 gives an interesting Fourier transform pair:

$$\sum_{n=-\infty}^{\infty} \delta\left(t - nT_0\right) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{j2\pi(kf_0)t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \delta(f - kf_0) \tag{41}$$



A special case when $T_0 = 1$ is quite easy to remember:

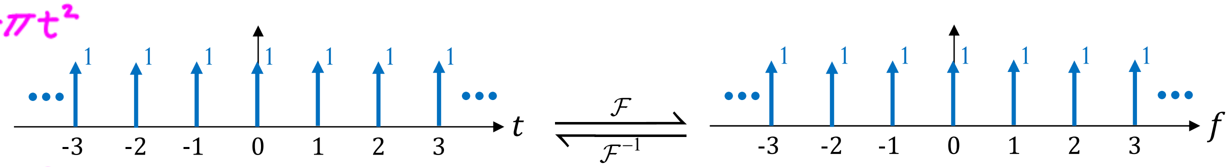
$$\sum_{n=-\infty}^{\infty} \delta\left(t - n\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \sum_{k=-\infty}^{\infty} \delta\left(f - k\right) \tag{42}$$

Another function that has this property is

$g(t) = e^{-\pi t^2}$

$\big\downarrow \mathcal{F}$

$G(f) = e^{-\pi f^2}$



Once we remember (42), we can easily use the scaling properties of the Fourier transform (21) and the delta function (18) to generalize the special case (42) back to (41):

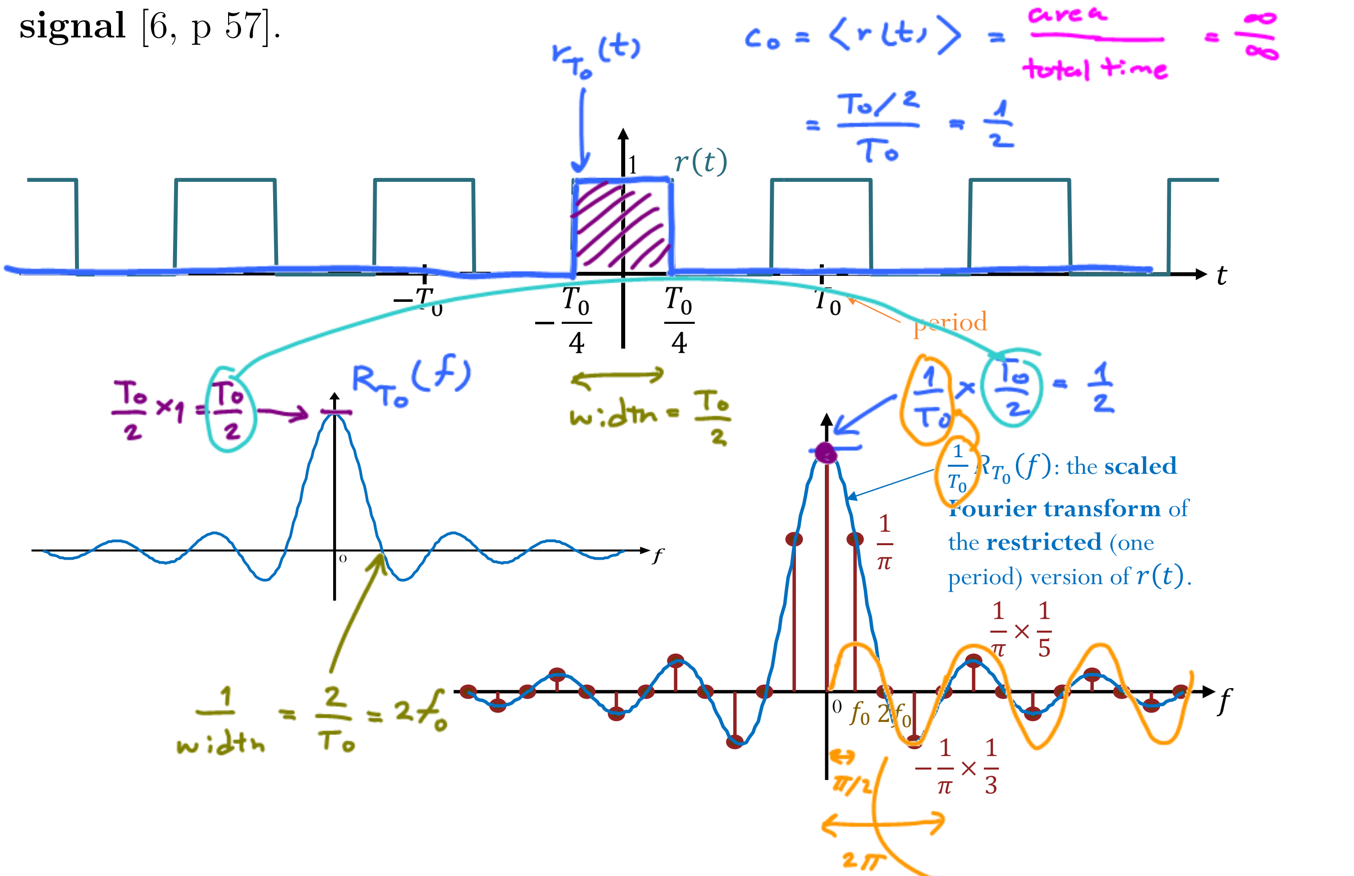
$$\sum_{n=-\infty}^{\infty} \delta\left(at - n\right) = x\left(at\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{|a|} X\left(\frac{f}{a}\right) = \frac{1}{|a|} \sum_{k=-\infty}^{\infty} \delta\left(\frac{f}{a} - k\right)$$

$$\frac{1}{|a|} \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{a}\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{|a|} |a| \sum_{k=-\infty}^{\infty} \delta\left(f - ka\right)$$

$$\sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{a}\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} |a| \sum_{k=-\infty}^{\infty} \delta\left(f - ka\right)$$

At the end, we plug-in $a = f_0 = 1/T_0$.

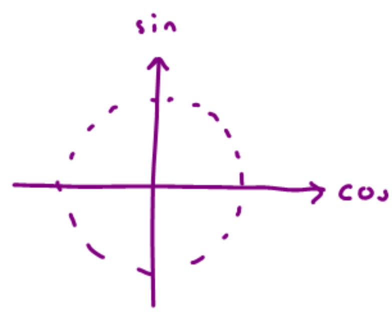
Example 4.47. Find the Fourier coefficients of the **square pulse periodic signal** [6, p 57].



$$c_k = \frac{1}{T_0}R_{T_0}(kf_0) = \frac{1}{T_0}\left(\frac{T_0}{2}\operatorname{sinc}\left(2\pi\left(\frac{T_0}{4}\right)(f)\right)\Bigg|_{f=kf_0}\right) = \frac{1}{2}\operatorname{sinc}\left(k\frac{\pi}{2}\right)$$

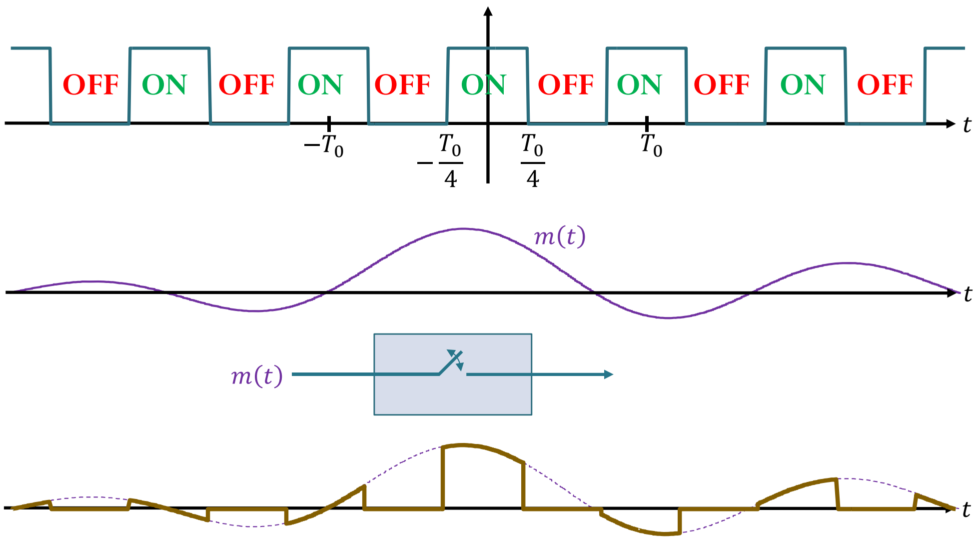
$$= \frac{1}{2}\frac{\sin\left(k\frac{\pi}{2}\right)}{k\frac{\pi}{2}} = \frac{\sin\left(k\frac{\pi}{2}\right)}{k\pi}$$

k	$k \times \frac{\pi}{2}$	$\sin\left(k \times \frac{\pi}{2}\right)$	$c_k = \frac{\sin\left(k\frac{\pi}{2}\right)}{k\pi}$	k
0		0	1/2	
1		1	$\frac{1}{\pi}$	−1
2		0	0	−2
3		−1	$\frac{-1}{3\pi}$	−3
4		0	0	−4
5		1	$\frac{1}{5\pi}$	−5
		0		
		−1		
		⋮		



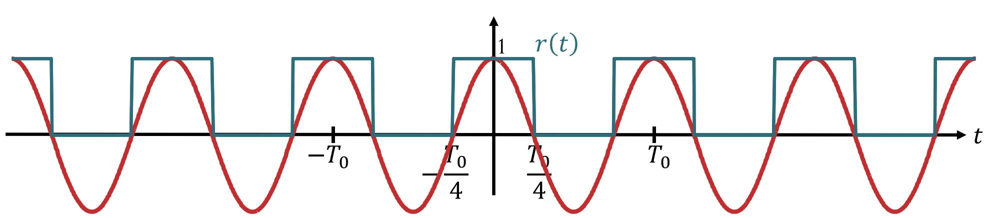
Remarks:

(a) Multiplication by this signal is equivalent to a switching (ON-OFF) operation. (Same as periodically turning the switch on (letting another signal pass through) for half a period T_0.



(b) This signal can be expressed via a cosine function with the same period:

$$r(t) = 1\left[\cos(2\pi f_0 t) \geq 0\right] = \begin{cases} 1, & \cos(2\pi f_0 t) \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$



(c) A **duty cycle** is the percentage of one period in which a signal is "active". Here,

$$\text{duty cycle} = \text{proportion of the "ON" time} = \frac{\text{width}}{\text{period}}.$$

In this example, the duty cycle is $\frac{T_0/2}{T_0} = 50\%$. When the duty cycle is $\frac{1}{n}$, the nth harmonic (c_n) along with its nonzero multiples are suppressed.

4.48. Parseval's Identity: $P_r = \left\langle |r(t)|^2 \right\rangle = \frac{1}{T_0}\int\limits_{T_0} |r(t)|^2\,dt = \sum\limits_{k=-\infty}^{\infty} |c_k|^2.$

4.49. Fourier series expansion for real valued function: Suppose $r(t)$ in the previous section is real-valued; that is $r^* = r$. Then, we have $c_{-k} = c_k^*$ and we provide here three alternative ways to represent the Fourier series expansion:

$$\tilde{r}(t) = \sum_{k=-\infty}^{\infty} c_n e^{j2\pi k f_0 t} = c_0 + \sum_{k=1}^{\infty}\left(c_k e^{j2\pi k f_0 t} + c_{-k}e^{-j2\pi k f_0 t}\right) \tag{43a}$$

=0 when r(t) is even

=0 when r(t) is odd

$$= c_0 + \sum_{k=1}^{\infty}\left(a_k \cos\left(2\pi k f_0 t\right)\right) + \sum_{k=1}^{\infty}\left(b_k \sin\left(2\pi k f_0 t\right)\right) \tag{43b}$$

$$= c_0 + 2\sum_{k=1}^{\infty}|c_k|\cos\left(2\pi k f_0 t + \angle c_k\right) \tag{43c}$$

where the corresponding coefficients are obtained from

$$c_k = \frac{1}{T_0}\int\limits_{\alpha}^{\alpha+T_0} r(t)\, e^{-j2\pi k f_0 t}dt = \frac{1}{2}\left(a_k - jb_k\right) \tag{44}$$

$$a_k = 2\mathrm{Re}\left\{c_k\right\} = \frac{2}{T_0}\int\limits_{T_0} r(t)\cos\left(2\pi k f_0 t\right)dt \tag{45}$$

$$b_k = -2\mathrm{Im}\left\{c_k\right\} = \frac{2}{T_0}\int\limits_{T_0} r(t)\sin\left(2\pi k f_0 t\right)dt \tag{46}$$

$$2|c_k| = \sqrt{a_k^2 + b_k^2} \tag{47}$$

$$\angle c_k = -\arctan\left(\frac{b_k}{a_k}\right) \tag{48}$$

$$c_0 = \frac{a_0}{2} \tag{49}$$

The Parseval's identity can then be expressed as

$$P_r = \left\langle |r(t)|^2 \right\rangle = \frac{1}{T_0}\int\limits_{T_0} |r(t)|^2 dt = \sum_{k=-\infty}^{\infty}|c_k|^2 = c_0^2 + 2\sum_{k=1}^{\infty}|c_k|^2$$

4.50. To go from (43a) to (43b) and (43c), note that when we replace c_{-k} by c_k^*, we have

$$\begin{aligned} c_k e^{j2\pi k f_0 t} + c_{-k} e^{-j2\pi k f_0 t} &= c_k e^{j2\pi k f_0 t} + c_k^* e^{-j2\pi k f_0 t} \\ &= c_k e^{j2\pi k f_0 t} + \left(c_k e^{j2\pi k f_0 t}\right)^* \\ &= 2\,\mathrm{Re}\left\{c_k e^{j2\pi k f_0 t}\right\}. \end{aligned}$$

- Expression (43c) then follows directly from the phasor concept:

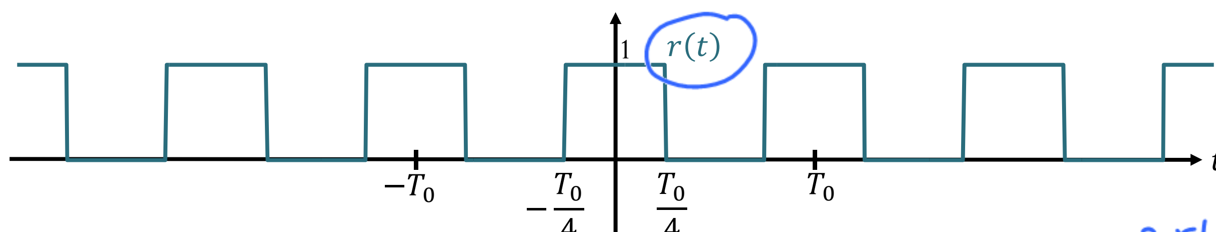
$$\mathrm{Re}\left\{c_k e^{j2\pi k f_0 t}\right\} = |c_k| \cos\left(2\pi k f_0 t + \angle c_k\right).$$

- To get (43b), substitute c_k by $\mathrm{Re}\{c_k\} + j\,\mathrm{Im}\{c_k\}$ and $e^{j2\pi k f_0 t}$ by $\cos(2\pi k f_0 t) + j \sin(2\pi k f_0 t)$.

Example 4.51. For the train of impulses in Example 4.44,

$$\sum_{n=-\infty}^{\infty} \delta(t-n) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{j2\pi(k f_0)t} = \frac{1}{T_0} + \frac{2}{T_0} \sum_{k=1}^{\infty} \cos k\omega_0 t \qquad (50)$$

Example 4.52. For the rectangular pulse train in Example 4.46,



Fourier series expansion:

$$r(t) = \frac{1}{2} + \frac{1}{\pi} e^{j(2\pi f_0 t)} - \frac{1}{3\pi} e^{j(2\pi(3f_0)t)} + \frac{1}{5\pi} e^{j(2\pi(5f_0)t)} + \cdots$$
$$+ \frac{1}{\pi} e^{j(2\pi(-f_0)t)} - \frac{1}{3\pi} e^{j(2\pi(-3f_0)t)} + \frac{1}{5\pi} e^{j(2\pi(-5f_0)t)} + \cdots$$

2 r(t)

Trigonometric Fourier series expansion:

$e^{jx} + e^{-jx} = 2\cos(x)$

$$r(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi f_0 t) - \frac{2}{3\pi}\cos(2\pi(3f_0)t) + \frac{2}{5\pi}\cos(2\pi(5f_0)t) + \cdots$$

$$1\left[\cos\omega_0 t \geq 0\right] = \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \dots\right) \tag{51}$$

Example 4.53. Bipolar square pulse periodic signal [6, p 59]:

$$\operatorname{sgn}(\cos\omega_0 t) = \frac{4}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \dots\right)$$

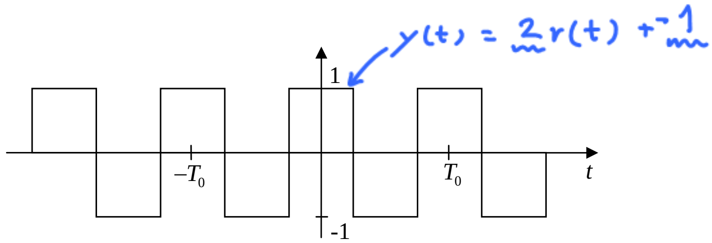


Figure 22: Bipolar square pulse periodic signal